Indian Statistical Institute
Semestral Examination
Differential Topology - MMath II

Max Marks: 60 Time: 180 minutes.

Give proper and complete justification(s) for your answers. Throughout $X, Y, \ldots$ are manifolds in some ambient euclidean space.

(1) (a) Show that $O(n)$, the set of all orthogonal matrices, is a manifold. Describe the tangent space at $A \in O(n)$. [8]

(b) Let X and Z be submanifolds of Y. Assume X and Z intersect transversally. Prove that if $x \in X \cap Z$, then

$$T_x(X \cap Z) = T_x(X) \cap T_x(Z).$$

[8]

(c) Suppose X, Z are submanifolds of Y that do not intersect transversally. May $X \cap Z$ still be a manifold? If so, what are the possibilities for $\operatorname{codim}(X \cap Z)$? [6]

(d) Let V be a vector space over $\mathbb{R}$ of dimension k. Show that whenever $\varphi_1, \ldots, \varphi_p \in V^*$ and $w_1, \ldots, w_p \in V$, then

$$\varphi_1 \wedge \cdots \wedge \varphi_p(w_1, \ldots, w_p) = \frac{1}{p!}\det\left(\varphi_i(w_j)\right).$$

[8]

(2) (a) Let ω be a nowhere zero 1-form on S^1. Show that ω cannot be exact. Is the same true for a nowhere zero 1-form on $\mathbb{R}$, $[0, 1]$? [2+2+2]

(b) For each p, $0 \leq p \leq 2$, construct a nowhere zero p-form on S^2. [8]

(c) Compute the integral

$$\int_C \omega$$

where $\omega = x\, dx - y\, dy$, $C = \operatorname{im}(\alpha)$ and $\alpha : [-1, 1] \longrightarrow \mathbb{R}^2$ is defined by $\alpha(t) = (t, t^2)$. Justify each step of your answer. [6]

(d) Show that $H^1_{dR}(\mathbb{R}^2 - 0) \neq 0$. [10]